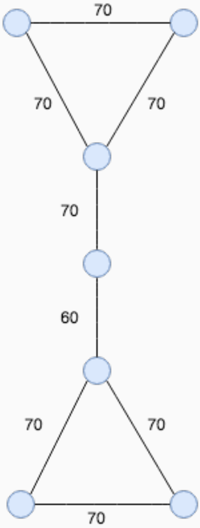
Matching Games

Computing the Nucleolus in Polynomial Time

By Justin Toth
Joint work with Jochen Könemann and Kanstantsin Pashkovich

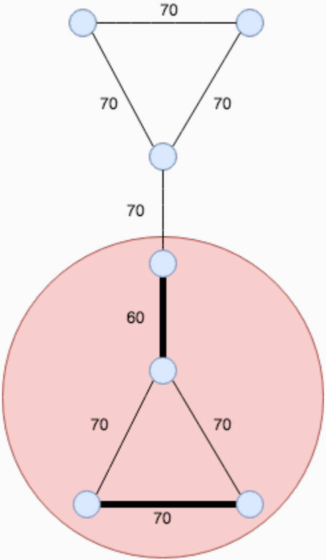
University of Waterloo

Weighted Matching Games



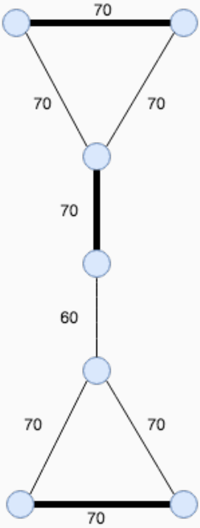
- Vertices: Players
- Edges: Potential Collaborations
- Have ≤ 1 collaborator
- Models Network Bargaining setting of [Kleinberg and Tardos '08]

Weighted Matching Games



- Value $\nu(S) = \max\{w(M) : M \text{ is a matching on } G[S]\}$
- $S \subseteq T \implies \nu(S) \leq \nu(T)$
- So assume the grand coalition forms.

Least Core



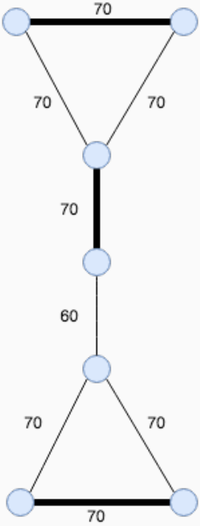
How to fairly distribute matching value to the players?

$$\sum_{i \in S} p_i \geq \nu(S) \qquad \forall S \subseteq V$$

$$\sum_{i \in V} p_i = \nu(V)$$

$$p_i \geq 0 \qquad \forall i \in V$$

Least Core



$$\nu(V) = 70 + 70 + 70 = 210$$
$$\nu_{\text{frac}}(V) = 7 \cdot 35 = 245$$

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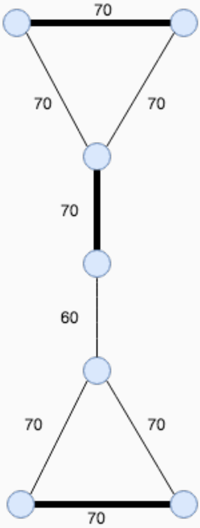

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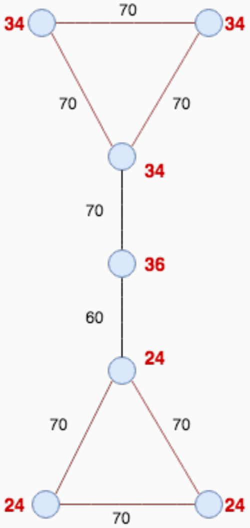
$$\begin{aligned} & \min \ \epsilon \\ \text{s.t.} \ & \sum_{i \in S} p_i \geq \nu(S) - \epsilon \quad && \forall S \subseteq V \\ & \sum_{i \in V} p_i = \nu(V) \\ & p_i \geq 0 && \forall i \in V \end{aligned}$$

An Outcome



$$\nu(V) = 210$$

$$\epsilon_1 = 70 - 2 \cdot 34 + 70 - 2 \cdot 24 = 24$$

$$\begin{aligned} & \min \epsilon \\ \text{s.t.} & \sum_{i \in S} p_i \geq \nu(S) - \epsilon \quad \forall S \subseteq V \\ & \sum_{i \in V} p_i = \nu(V) \\ & p_i \geq 0 \quad \forall i \in V \end{aligned}$$

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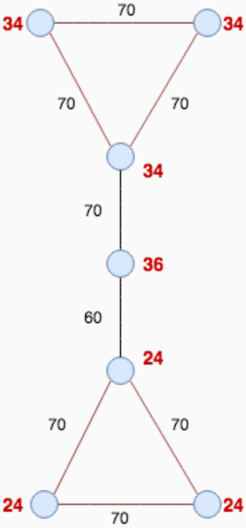

Problems with the Least Core:

- outcomes are not unique
- Excess: $p(S) - \nu(S)$ only controlled for worst coalitions

The Nucleolus

The *nucleolus* is *fairest* allocation p^*

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Sort excesses: $ex_p(S_1) \geq ex_p(S_2) \geq \cdots \geq ex_p(S) \geq \cdots =: \theta(p)$

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p^* lexicographically minimizes $\theta(p)$

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p^* lexicographically minimizes $\theta(p)$

And it is **unique**!

Hardness of Computing the Nucleolus

- Shortest Path Games
- Node-Weighted Matching
- Convex Games
- Spanning Tree Games
- Flow Games
- Weighted Voting Games
- Linear Production Games

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In P

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NP-hard

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Open for 15 years: How about Weighted Matching Games?

[Kern and Paulusma '03]

Hardness of Computing the Nucleolus

In P

- Shortest Path Games
- Node-Weighted Matching
- Convex Games
- **Our Result:** Matching Games

NP-hard

- Spanning Tree Games
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Open for 15 years: How about Weighted Matching Games?

[Kern and Paulusma '03]

Maschler's Scheme

Solve a hierarchy of linear programs

$$P_1 \supseteq P_2 \supseteq \cdots \supseteq P_{|V|} \equiv p^*$$

Subsequent LPs minimize excess over coalitions not fixed

$$\begin{aligned} P_2 : \min\ & \epsilon \\ \text{s.t. }\ & p \in P_1 \\ & p(S) \geq \nu(S) - \epsilon & \forall S \notin Fix(P_1) \end{aligned}$$

$$Fix(P_1) = \{S \subset V : \exists c \text{ s.t. } p(S) = c, \ \ \forall p \in P_1\}$$

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Each LP fixes new coalitions $\implies$ dimension of P_{i+1} reduces

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$O(2^{|V|})$ constraints in naive formulation of each P_i

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To implement Maschler's Scheme give a polynomial sized formulation for each P_i

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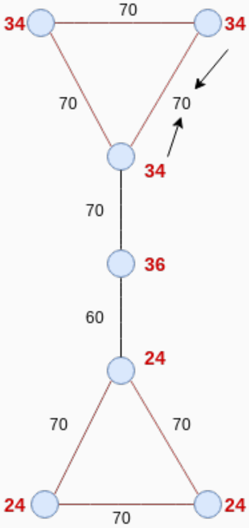
Separation for P_1



$$\min \epsilon$$

$$\text{s.t.} \sum_{i \in S} p_i \geq \nu(S) - \epsilon \qquad \forall S \subseteq V \text{ (???)}$$

$$\sum_{i \in V} p_i = \nu(V) \qquad \text{(Easy!)}$$

$$p_i \geq 0 \qquad \forall i \in V \text{ (Easy!)}$$

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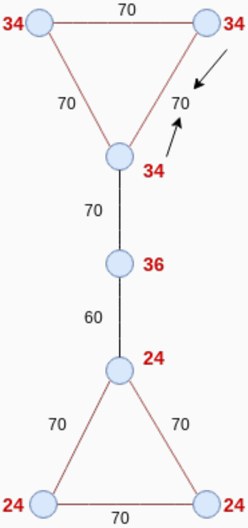

$$\overline{w}(uv) = w(uv) - p_u - p_v$$

$$\begin{aligned}
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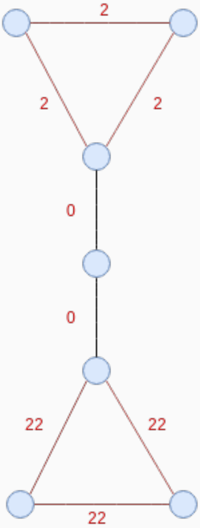

$\overline{w}(uv) = w(uv) - p_u - p_v$ Min

Excess Matchings: $\max \overline{w}(M)$

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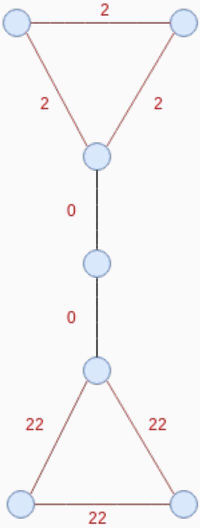

$\overline{w}(uv) = w(uv) - p_u - p_v$ Min

Excess Matchings: $\max \overline{w}(M)$

Solve using your favourite max weight matching algorithm!

$$
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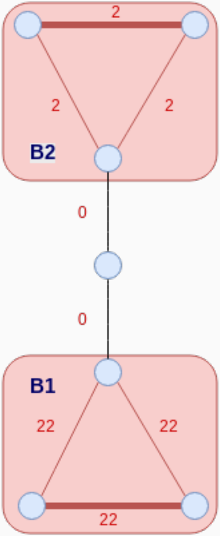
Describing Constraints of P_1



Tight P_1 constraints $\equiv$ optimal $\overline{w}$ matchings

Optimal Matchings defined by:

$$x(E(B)) = \frac{|B|-1}{2} \quad \forall B \in \mathcal{L}$$

$$x(\delta(v)) = 1 \quad \forall v \in T$$

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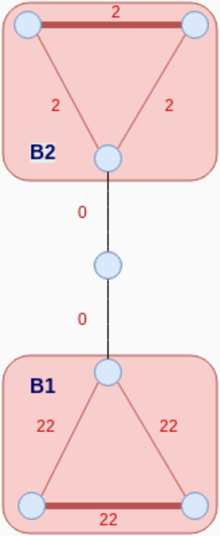

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Can show $T = \emptyset$ when $\epsilon_1 > 0$

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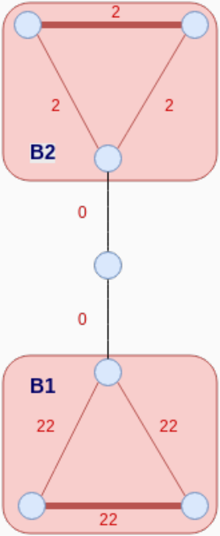

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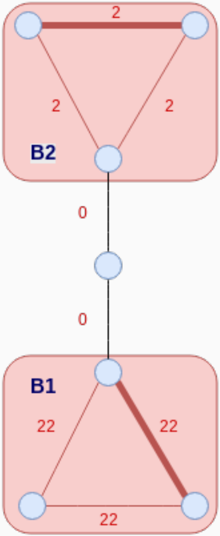

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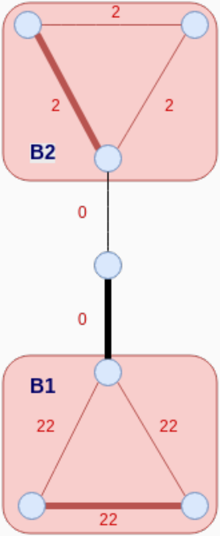

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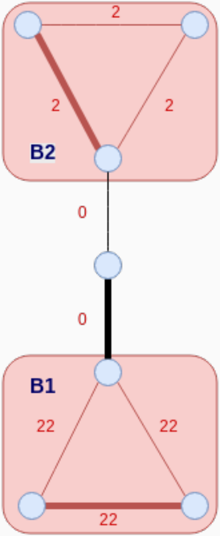

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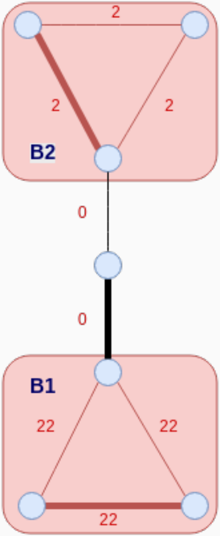

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Need something for all $p \in P_1$

Universal Matchings

If $\overline{p}$ is in relative interior of P_1 we call $\overline{p}$ a universal allocation.

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Studying the structure of universal matchings will be key!

Three Ideas for Description

Consider laminar blossom family $\mathcal{L}$ for $\overline{p}$

(Recall: Matching M is optimal $\iff |M \cap E(B)| = \frac{|B|-1}{2} \quad \forall B \in \mathcal{L}$)

Three Ideas for Description

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Observations

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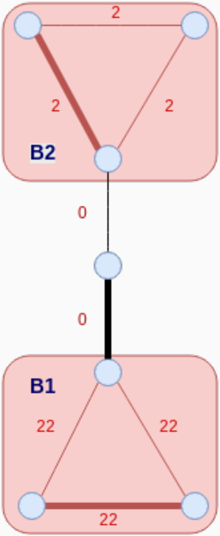
Theorem: These linear constraints suffice to describe Least Core P_1

Observations 1 and 2



1. $\overline{w}$ values > 0 on blossom edges.

Proof Idea: Drop non-positive edge e from M: Optimal but $|M - e \cap E(B)| < |M \cap E(B)| = \frac{|B|-1}{2}$

■

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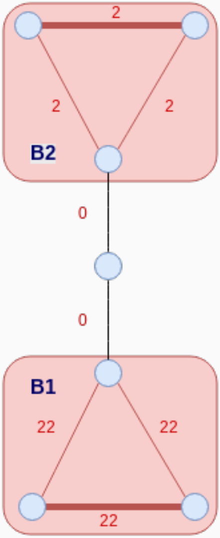

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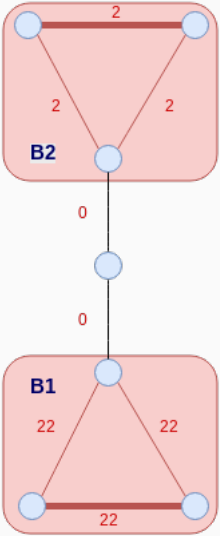
Proof Idea: Drop non-positive edge e from M: Optimal but $|M - e \cap E(B)| < |M \cap E(B)| = \frac{|B|-1}{2}$ ■

2. $\overline{w}$ values ≤ 0 on non-blossom edges.

Proof Idea: Can satisfy $\mathcal{L}$ constraints without such edges. ■

Observation 3



Need to understand $\overline{w}$ distribution over each $B \in \mathcal{L}$

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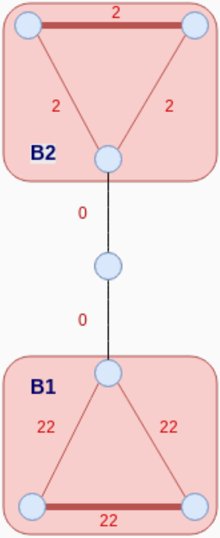

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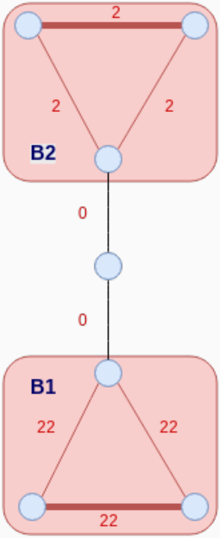

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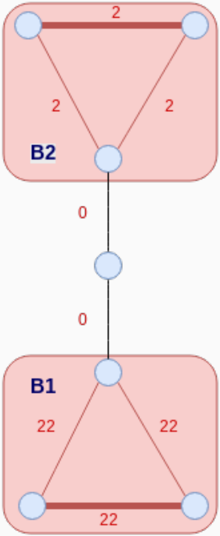
Fails for larger examples!

Observation 3



Actually: $\forall u, v \in B, \forall B \in \mathcal{L}$

$$p_u - \overline{p}_u = p_v - \overline{p}_v$$

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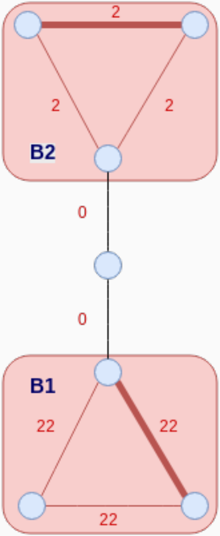

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$$p_u - \overline{p}_u = p_v - \overline{p}_v$$

Technical Proof. **Key Idea:** M exposing u can be modified within B to expose v.

Least Core Description

Obtain a nice P_1 description

$$\begin{aligned} \min \quad & \epsilon \\ \text{s.t.} \quad p_v - \overline{p}_v &= p_u - \overline{p}_u && \text{for all} \quad u, v \in S_i^*,\ i \in [k] \\ p(e) - w(e) &\leq 0 && \text{for all} \quad e \in E^* \\ p(e) - w(e) &\geq 0 && \text{for all} \quad e \in E^+ \\ p(M^*) - w(M^*) &= -\epsilon && M^* \text{ a universal matching} \\ p(V) &= \nu(G) \\ p &\geq 0 \end{aligned}$$

Where $S_1^*, \ldots, S_k^*$ are maximal sets of $\mathcal{L}$,

$E^* := \left(\bigcup_{i \in [k]} E(S_i^*)\right) \cap \left(\bigcup_{M \in \mathcal{M}^*} M\right)$,

and E^+ is the edges with at most one endpoint each S_i^*.

Implementing Maschler's Scheme

Solve a hierarchy of linear programs

$$P_1 \supseteq P_2 \supseteq \cdots \supseteq P_{|V|} \equiv p^*$$

To find nucleolus$= p^*$ give a polynomial sized formulation for each P_i

$$\begin{aligned} P_2 : \min\ & \epsilon \\ \text{s.t. } & p \in P_1 \\ & p(S) \geq \nu(S) - \epsilon \qquad \forall S \notin Fix(P_1) \end{aligned}$$

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$$\begin{aligned} P_2 : \min\ & \epsilon \\ \text{s.t. } & p \in P_1 \\ p(e) - w(e) &\geq \epsilon - \epsilon_1 && \forall e \in E^+, e \notin Fix(P_1) \\ p_v &\geq \epsilon - \epsilon_1 && \forall v \notin Fix(P_1) \\ p(e) - w(e) &\leq \epsilon - \epsilon_1 && \forall e \in E^*, e \notin Fix(P_1) \end{aligned}$$

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Implementing Maschler's Scheme

Observation 3 (Blossom Symmetry) plays a major role.

$$\forall u, v \in B, \forall B \in \mathcal{L}, p_u - \overline{p}_u = p_v - \overline{p}_v$$

Lemma: $B \in \mathcal{L} \setminus Fix(P_1) \implies v \notin Fix(P_1) \quad \forall v \in B$

$$\begin{aligned} P_2 : \min\ & \epsilon \\ \text{s.t. } & p \in P_1 \\ p(e) - w(e) \geq \epsilon - \epsilon_1 & \qquad \forall e \in E^+, e \notin Fix(P_1) \\ p_v \geq \epsilon - \epsilon_1 & \qquad \forall v \notin Fix(P_1) \\ p(e) - w(e) \leq \epsilon - \epsilon_1 & \qquad \forall e \in E^*, e \notin Fix(P_1) \end{aligned}$$

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