# Matching Games

## Computing the Nucleolus in Polynomial Time

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Joint work with Jochen Könemann and Kanstantsin Pashkovich

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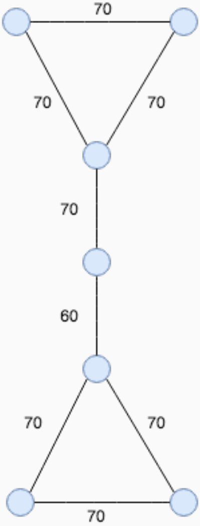
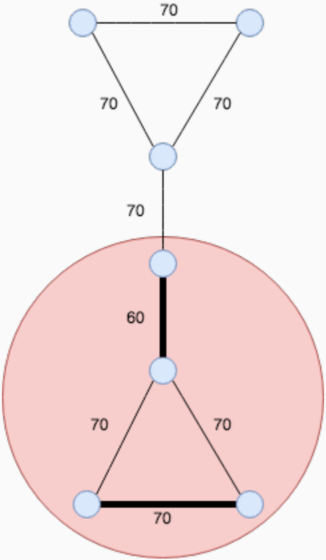
# Weighted Matching Games



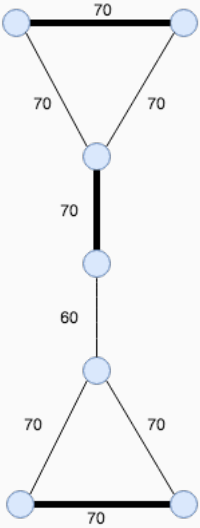

- Vertices: Players
- Edges: Potential Collaborations
- Have $\leq$ one collaborator
- Models Network Bargaining setting of [Kleinberg and Tardos '08]

# Weighted Matching Games




- Value $\nu(S) = \max\{w(M) : M$ is a matching on $G[S]\}$
- $S \subseteq T \implies \nu(S) \leq \nu(T)$
- So assume the grand coalition forms.

# Least Core



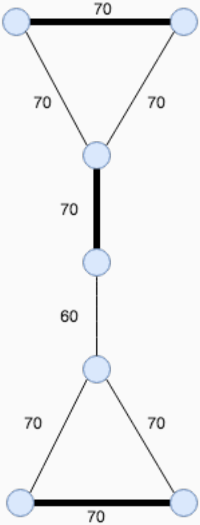

How to fairly distribute matching value to the players?

$$\sum_{i \in S} p_i \geq \nu(S) \qquad \forall S \subseteq V$$

$$\sum_{i \in V} p_i = \nu(V)$$

$$p_i \geq 0 \qquad \forall i \in V$$

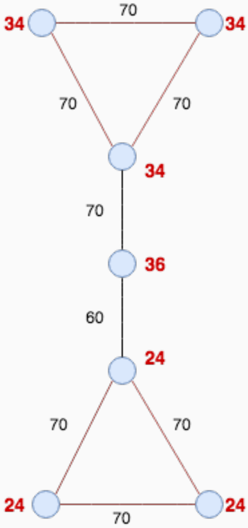
# Least Core




$$\nu(V) = 70 + 70 + 70 = 210$$

$$\begin{aligned} &\min \ \epsilon \\ \text{s.t.} \ &\sum_{i \in S} p_i \geq \nu(S) - \epsilon \quad && \forall S \subseteq V \\ &\sum_{i \in V} p_i = \nu(V) \\ &p_i \geq 0 && \forall i \in V \end{aligned}$$

# An Outcome




$$\nu(V) = 210$$

$$\epsilon_1 = 70 - 2{\cdot}34 + 70 - 2 \cdot 24 = 24$$

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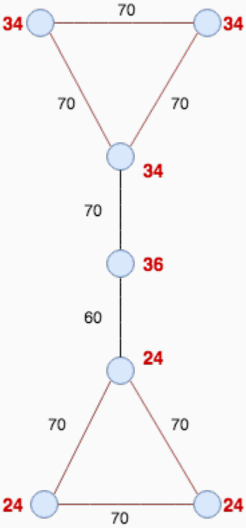


Problems with the Least Core:

- outcomes are not unique
- Excess: $p(S) - \nu(S)$ only controlled for worst coalitions

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And it is **unique**!

# Hardness of Computing the Nucleolus

- Shortest Path Games
- Node-Weighted Matching
- Convex Games
- Spanning Tree Games
- Flow Games
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Open for over 10 years: How about Weighted Matching Games?

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- **Our Result:** Matching Games

NP-hard

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# Maschler's Scheme

Solve a hierarchy of linear programs

$$P_1 \supseteq P_2 \supseteq \cdots \supseteq P_{|V|}$$

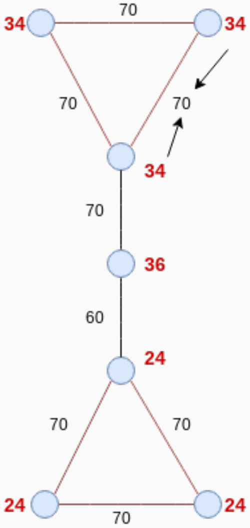
Subsequent LPs minimize excess over coalitions not fixed

Each LP fixes new coalitions $\implies$ dimension reduces

Exponential size families of constraints in naive formulation

Study structure of coalitions with max excess $\implies$ polynomial size formulations.

# Universal Matchings




$$\bar{w}(uv) = w(uv) - p_u - p_v$$

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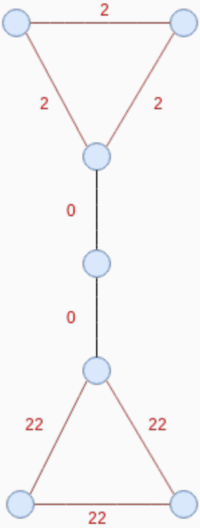


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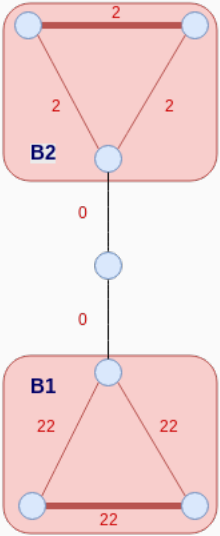


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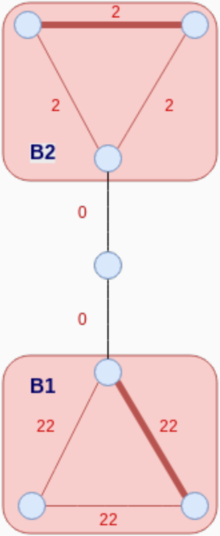


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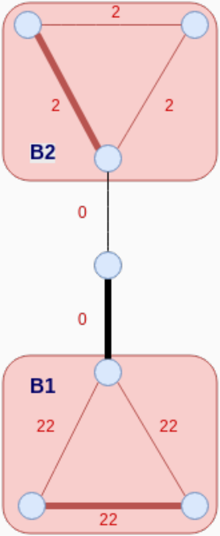


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Find $p$ with universal optimal matchings?

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Lemma: $\exists$ allocation $p^*$ whose optimal matchings are universal.

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Theorem: These linear constraints are sufficient to describe Least Core

# Least Core Description

Obtain a nice least core description

$$
\begin{array}{lrcll}
\min & \epsilon & & & \\
\text{s.t.} & p(v) - p^*(v) & = & p(u) - p^*(u) & \text{for all} \quad v, u \in S_i^*,\ i \in [k] \\
 & p(e) - w(e) & \leq & 0 & \text{for all} \quad e \in E^* \\
 & p(e) - w(e) & \geq & 0 & \text{for all} \quad e \in E^+ \\
 & p(M^*) - w(M^*) & = & -\epsilon & M^* \text{ a universal matching} \\
 & p(V) & = & \nu(G) & \\
 & p & \geq & 0 &
\end{array}
$$

Where $S_1^*, \ldots, S_k^*$ are maximal sets of $\mathcal{L}$,

$E^* := \left(\bigcup_{i \in [k]} E(S_i^*)\right) \cap \left(\bigcup_{M \in \mathcal{M}^*} M\right)$,

and $E^+$ is the edges with at most one endpoint each $S_i^*$.